Journée fédérations MIRES/MARGAUx (Poitiers)
EDP : fiabilité/vérification de codes, étude de convergence

Analyse numérique en Rocq
Vers une formalisation de la méthode des éléments finis
Partie II : Les éléments finis de Lagrange simpliciaux

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Inria, LIPN - USPN (U. Paris 13)

19/11/2025

Formalization Process (1/2) — Use Literature

Formalization of mathematics

⇝ translate existing literature

Textbooks used for rocq-num-analysis

Undergraduate Mathematics:	B. Gostiaux. *1. Algèbre. 2. Topologie, analyse réelle. 3. Analyse fonctionnelle et calcul diff.* PUF, 1993
Lax–Milgram:	H. Brezis. *Analyse fonctionnelle*. Masson, 1983 P.G. Ciarlet. *The Finite Element Method for Elliptic Problems*. SIAM, 2002 A. Ern, J.-L. Guermond. *Theory and Practice of Finite Elements*. Springer, 2004
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Detailed *pen-and-paper* proofs for rocq-num-analysis

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+ make formalization design choices

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Formalization Process (2/2) — Feedback

Feedback: formal proof allows another view of mathematics
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In practice

- choose proof paths mostly using already formalized notions
  $\rightsquigarrow$ e.g. Lax–Milgram rather than Banach–Nečas–Babuška
  $\rightsquigarrow$ e.g. exploit affine properties for Lagrange FE
- try to generalize results (with reasonable efforts)
- try to choose constructive proof paths (with reasonable efforts)

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Teaching purposes: accessibility rather than generalization

Another View of Mathematics

Factorization of results via generalization/abstraction
e.g. modules on a ring of scalars vs vector spaces on a field

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Avoid unnecessary hypotheses in definitions:

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  e.g. minus on $\mathbb{N}$ (0), reciprocal on $\mathbb{R}$ (0),
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```
Context {T1 T2 : Type}.                          (* unit type means singleton *)
Lemma fun_from_empty_is_unit : ¬ inhabited T1 → is_unit_type (T1 → T2).
Lemma fun_to_empty_is_empty :
  inhabited T1 → ¬ inhabited T2 → ¬ inhabited (T1 → T2).
```

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⇝ $\mathbb{R}^n$ is represented by $[0..n) \to \mathbb{R}$, thus $\mathbb{R}^0 = \{[]\} = \{0\}$
dim = 0, i.e. the empty family of empty vectors is a basis

Outline

Set Systems (1/3) — Elementary Properties

FC, VM. RR-9386 (v3), chap. 8, pp. 107–148. (WIP: rocq-num-analysis branch Subset)

Definition $P \subset \mathscr{P}(X)$ + closedness properties (e.g. under ^{c}, or $\bigcup_{\mathbb{N}}$)
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```
Notation subset X := (X → Prop).                    (* belonging function *)
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Notation subset X := (X → Prop).                          (* belonging function *)

Context {T : Type}.

(* operations on sets *)
Definition emptyset : subset T := fun _ ⇒ False.
Definition compl (A : subset T) : subset T := fun x ⇒ ¬ A x.
Definition union_seq (As : ℕ → subset T) : subset T := fun x ⇒ ∃ n, As n x.
```

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Notation subset_system X := (subset (subset X)).    (* (X → Prop) → Prop *)
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(* closedness properties of set systems *)
Context (P : subset_system T).
Definition wEmpty : Prop := P emptyset.                                    (* wE *)
Definition Compl : Prop := ∀ A, P A → P (compl A).                          (* C *)
Definition Union_seq : Prop := ∀ As, (∀ n, P (As n)) → P (union_seq As). (* Uc *)
```

Set Systems (2/3) — Constructive Approach

Generated σ-algebra: $G \subset \mathscr{P}(X)$, $\Sigma(G) \overset{\text{def}}{=} \bigcap_{P\,\sigma-\text{alg},\,G \subset P} P$

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Context {T : Type} (gen : subset_system T).
Inductive Sigma_algebra : subset_system T :=
  | Sigma_algebra_Gen : Incl gen Sigma_algebra
  | Sigma_algebra_wEmpty : wEmpty Sigma_algebra                  (* wE *)
  | Sigma_algebra_Compl : Compl Sigma_algebra                     (* C *)
  | Sigma_algebra_Union_seq : Union_seq Sigma_algebra.           (* Uc *)
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$\rightsquigarrow$ proofs by induction for properties on σ-algebras (only 4 cases)

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Usual predicate form

```
Definition is_Sigma_algebra P : Prop := Sigma_algebra P = P.
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Definition is_Sigma_algebra P : Prop := Sigma_algebra P = P.
Lemma is_Sigma_algebra_equiv :                         (* Σ ⇔ wE ∧ C ∧ Uc *)
  ∀ P, is_Sigma_algebra P ↔ wEmpty P ∧ Compl P ∧ Union_seq P.
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```

⇝ π/λ-system, ring/algebra of sets, monotone class and σ-ring
Semiring and semi-algebra remain in predicate form only

Set Systems (3/3) — High-level Proofs

```
Theorem monotone_class :
  Sigma_algebra (Algebra gen) = Monotone_class (Algebra gen).
Proof.
rewrite ←Sigma_algebra_Monotone_class.
apply Algebra_Monotone_class_is_Sigma_algebra.
+ apply Monotone_class_Algebra_is_Algebra.
+ apply Monotone_class_idem.
Qed.
```

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1: Sigma_algebra (Algebra gen) = Monotone_class (Algebra gen)
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```
1: Sigma_algebra (Algebra gen) = Monotone_class (Algebra gen)
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```
Check Sigma_algebra_Monotone_class.
```

$\forall$ `G, Sigma_algebra (Monotone_class G) = Sigma_algebra G`

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```
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```

```
Check Algebra_Monotone_class_is_Sigma_algebra.
                ∀ G, is_Algebra G → is_Monotone_class G → is_Sigma_algebra G
```

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Goals

```
1: is_Algebra (Monotone_class (Algebra gen))
2: is_Monotone_class (Monotone_class (Algebra gen))
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```
Check Algebra_Monotone_class_is_Sigma_algebra.
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```
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                          ∀G, is_Algebra (Monotone_class (Algebra G))
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Set Systems (3/3) — High-level Proofs

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Set Systems (3/3) — High-level Proofs

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```

Goal

```
1: is_Monotone_class (Monotone_class (Algebra gen))
```

```
Check Monotone_class_idem.          ∀ G, is_Monotone_class (Monotone_class G)
```

Set Systems (3/3) — High-level Proofs

```
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Goal

```
All goals completed.
```

Set Systems (3/3) — High-level Proofs

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Similar proof path for Dynkin π-λ theorem

Outline

1. Introduction
2. About Measure Theory
    - Set Systems
    - Lebesgue Induction Principle
3. Simplicial Lagrange Finite Elements
4. Difficulties, Conclusion and Perspectives

Lebesgue Induction Principle (1/3) — Some Definitions

S. Boldo, FC, VM, MM, H. Mouhcine. *A Coq Formalization of Lebesgue Induction Principle and Tonelli's Theorem*.
Formal Methods 2023. (rocq-num-analysis v2.0)

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Measurability of functions

```
Context {T1 T2 : Type} (gen1 : subset_system T1) (gen2 : subset_system T2).

(* measurable is an alias for Sigma_algebra *)
Definition measurable_fun : subset (T1 → T2) :=        (* ∀A2 ∈ Σ2, f^-1(A2) ∈ Σ1 *)
  fun f ⇒ ∀ A2, measurable gen2 A2 → measurable gen1 (fun x1 ⇒ A2 (f x1)).
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  fun f ⇒ ∀ A2, measurable gen2 A2 → measurable gen1 (fun x1 ⇒ A2 (f x1)).
```

For extended real-valued functions

```
Context {T : Type} (gen : subset_system T).

(* with T1 := T and T2 := $\overline{\mathbb{R}}$ *)
Definition meas_fun_Rbar := measurable_fun gen gen_Rbar.
```

Lebesgue Induction Principle (1/3) — Some Definitions

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For extended real-valued functions

```
Context {T : Type} (gen : subset_system T).

(* with T1 := T and T2 := R̄ *)
Definition meas_fun_Rbar := measurable_fun gen gen_Rbar.
Definition Mplus : subset (T → Rbar) := fun f ⇒ nonneg f ∧ meas_fun_Rbar f.
```

Lebesgue Induction Principle (2/3) — Inductive Definition

Constructive view of $\mathscr{M}_+$

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Constructive view of $\mathscr{M}_+$

```
Context {T : Type} (gen : subset_system T).

Inductive Mp : subset (T → Rbar) :=
  | Mp_charac : ∀ A, measurable gen A → Mp (charac A)
  | Mp_scal : ∀ a f, 0 ≤ a → Mp f → Mp (fun x ⇒ a *_ℝ̄ f x)
  | Mp_plus : ∀ f g, Mp f → Mp g → Mp (fun x ⇒ f x +_ℝ̄ g x)
  | Mp_sup : ∀ fs, incr_fun_seq fs → (* fs n ≤ fs (S n) *)
                  (∀ n, Mp (fs n)) → Mp (fun x ⇒ Sup_seq (fun n ⇒ fs n x)).
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  | Mp_sup : ∀ fs, incr_fun_seq fs → (* fs n ≤ fs (S n) *)
               (∀ n, Mp (fs n)) → Mp (fun x ⇒ Sup_seq (fun n ⇒ fs n x)).

Lemma Mp_correct : ∀ f, Mp f ↔ Mplus gen f.
```

Lebesgue Induction Principle (3/3) — Induction Principle

```
Lemma Mp_ind : ∀ (P : subset (T → Rbar)),                 (* generated lemma *)
  (∀ A, measurable gen A → P (charac A)) →
  (∀ a f, 0 ≤ a → Mp f → P f → P (fun x ⇒ a *_R̄ f x)) →
  (∀ f g, Mp f → P f → Mp g → P g → P (fun x ⇒ f x +_R̄ g x)) →
  (∀ f, incr_fun_seq f →
    (∀ n, Mp (f n)) → (∀ n, P (f n)) → P (fun x ⇒ Sup_seq (fun n ⇒ f n x))) →
  ∀ f, Mp f → P f.
```

Lemma (Lebesgue induction principle)

Let P be a predicate on $X \to \overline{\mathbb{R}}$. Assume P holds on $\mathscr{IF}$ and is compatible on $\mathscr{M}_+$ with positive linear operations and with the supremum of nondecreasing sequences:

$$\begin{aligned}
\forall A \in \Sigma, &\quad P(\mathbb{1}_A),\\
\forall a \in \mathbb{R}_+, \forall f \in \mathscr{M}_+, &\quad P(f) \Rightarrow P(af),\\
\forall f, g \in \mathscr{M}_+, &\quad P(f) \wedge P(g) \Rightarrow P(f+g),\\
\forall (f_n)_{n\in\mathbb{N}} \in \mathscr{M}_+, &\quad (\forall n \in \mathbb{N},\, f_n \le f_{n+1} \wedge P(f_n)) \Rightarrow P\left(\sup_{n\in\mathbb{N}} f_n\right).
\end{aligned}$$

Then, P holds on $\mathscr{M}_+$.

Outline

Simplicial Lagrange Finite Elements

SB, FC, VM, MM, HM. *A Rocq Formalization of Simplicial Lagrange FE*. RR-9590, Inria Saclay, 2025. (rocq-num-analysis v2.0)

Finite element method

T_{geom} transfers computations on any cell K to the reference unit cell $\widehat{K}$

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$\rightsquigarrow$ do the same to build Lagrange FE
define quantities / prove properties in the reference case
then transport defs/most proofs using affine properties of T_{geom}

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B. Gostiaux (*4. Géométrie affine et métrique*. PUF, 1995)

> *Une propriété vraiment affine se traduit par des barycentres*
> *[A truly affine property translates into barycenters]*

Barycentric coordinates = affine Lagrange polynomials ($k = 1$)

Formalization of Finite Elements (1/2) — Definition

Finite element as triple (K, P, Σ) (Ciarlet, 1991)

Given $d, n_{\text{dof}} \geq 1$ and $q \in \{1, d\}$

Geometric element $K \subset \mathbb{R}^d$ polytope, $\text{int}(K) \neq \emptyset$ (e.g. d-simplex)

Approximation space $P \subset \mathscr{F}(K \to \mathbb{R}^q)$ vector space, $\dim \in \mathbb{N}^\star$ (e.g. poly)

Degrees of freedom $\Sigma = (\sigma_i)_{i \in [0..n_{\text{dof}})} \in \mathscr{F}(P, \mathbb{R})$ lin (e.g. $\sigma_i(p) = p(a_i)$)

unisolvence: $\Phi_\Sigma \overset{\text{def}}{=} (p \mapsto (\sigma_i(p))_{i \in [0..n_{\text{dof}})})$ bij onto $\mathbb{R}^{n_{\text{dof}}}$

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$$\text{unisolvence} \iff \dim P = n_{\text{dof}} \quad \wedge \quad \Phi_\Sigma \text{ inj}$$

Formalization of Finite Elements (2/2) — Rocq Record

v1: choose $q = 1$, K simplex

```
Record FE (d : ℕ) : Type := mk_FE {
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$\rightsquigarrow$ build the record for simplicial Lagrange FE

Outline

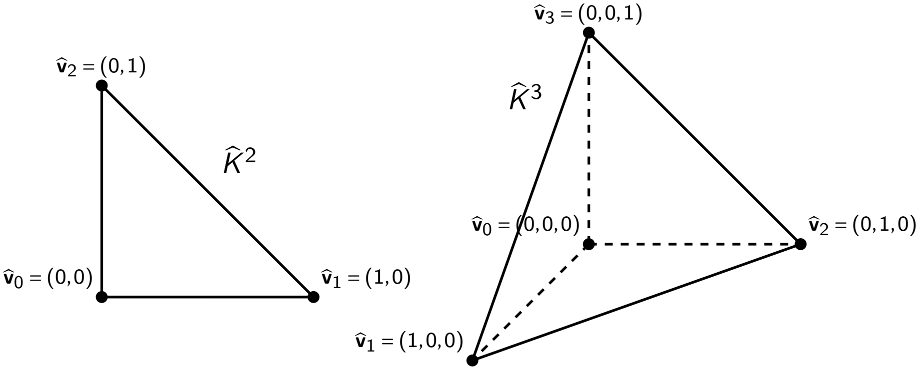
1. Introduction

2. About Measure Theory

3. Simplicial Lagrange Finite Elements
   - Reference Simplex
   - Generic Simplex
   - Lagrange Finite Element

4. Difficulties, Conclusion and Perspectives

Reference Simplex (1/3) — Vertices and Nodes



bold = vector/point/tuple
hat = reference

Reference vertices $\quad \widehat{\mathbf{v}} \overset{\text{def}}{=} (\mathbf{0}, \boldsymbol{\delta}_0, \ldots, \boldsymbol{\delta}_{d-1}) \qquad (\forall i \in [0..d), \boldsymbol{\delta}_i \overset{\text{def}}{=} (\delta_{ij})_{j \in [0..d)})$

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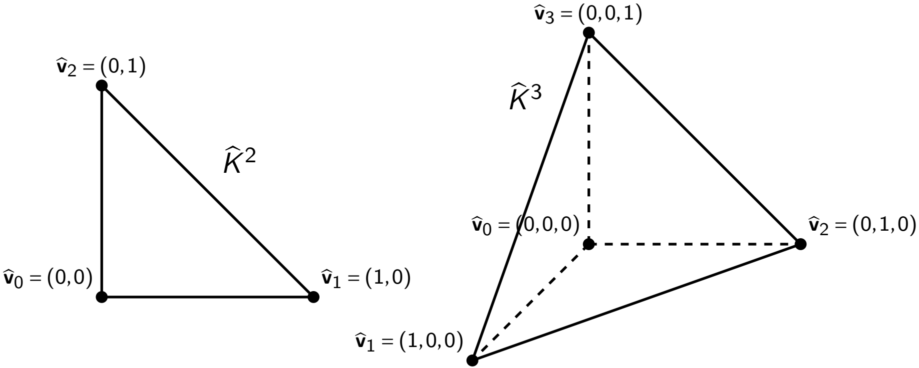

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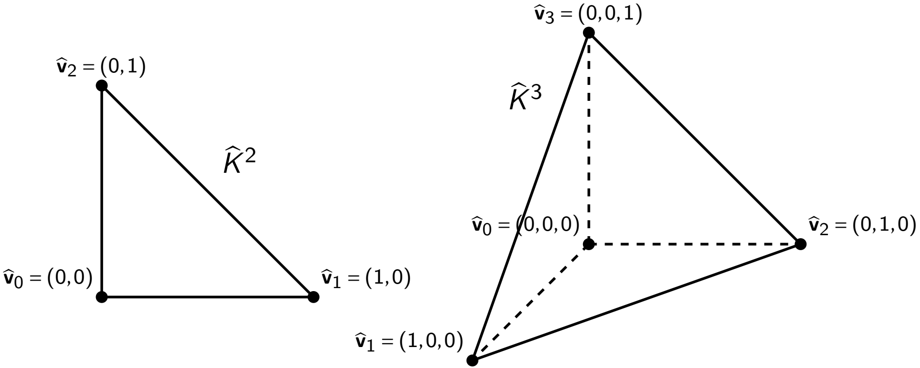

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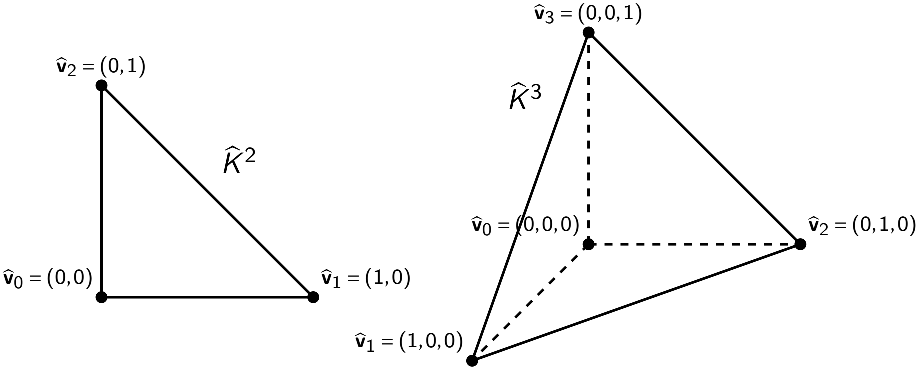

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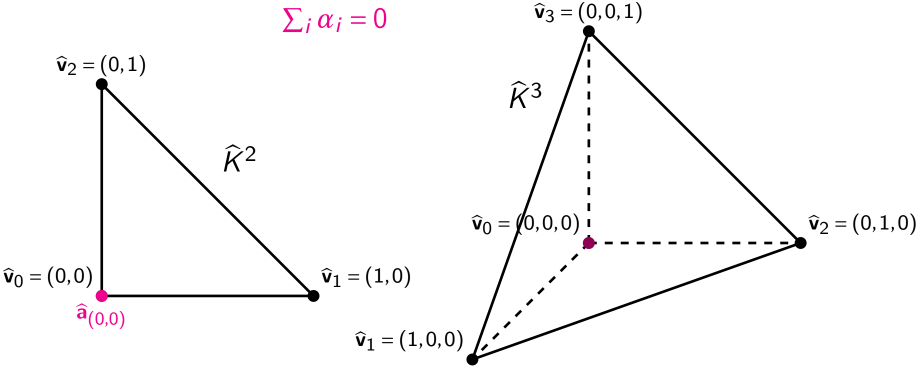

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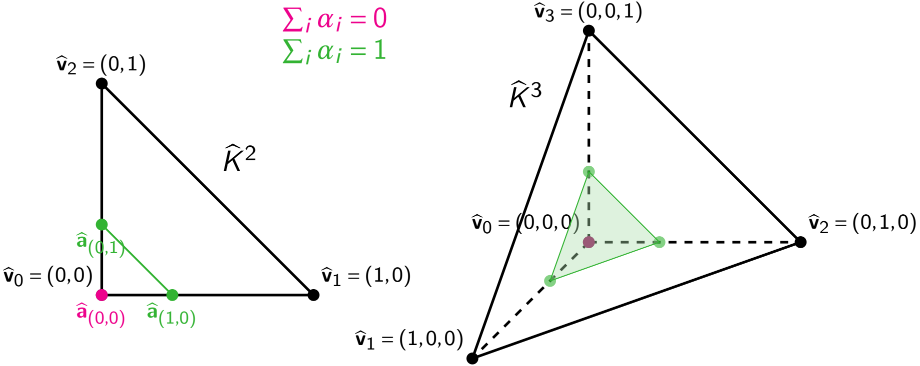

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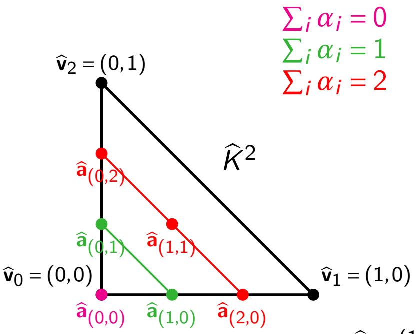

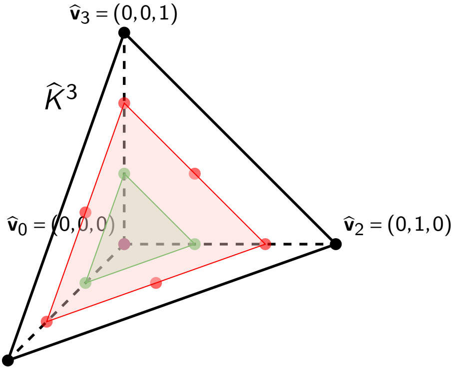

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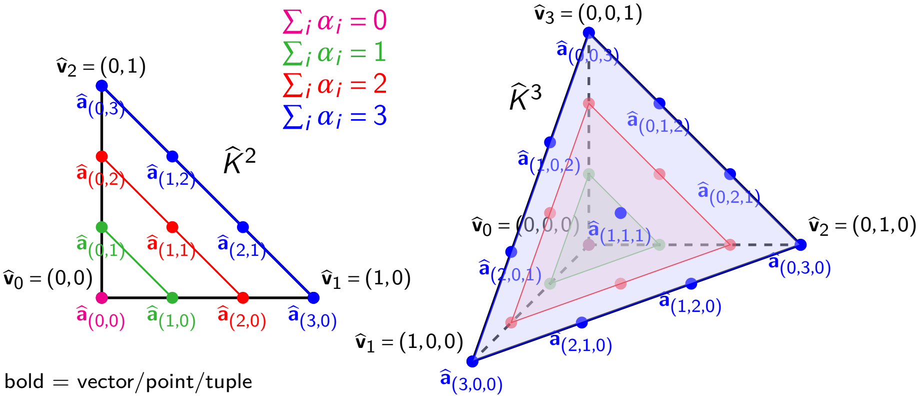

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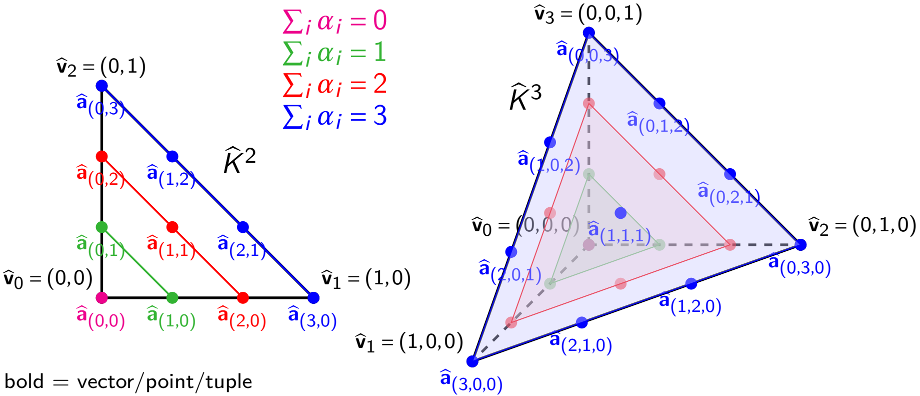

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$\rightsquigarrow$ $\widehat{\mathbf{v}}$ are ref nodes, $\qquad k = 1 \Rightarrow \widehat{\mathbf{a}} =$ ref vertices

Reference Simplex (2/3) — Lagrange Polynomials

Reference Lagrange polynomials given $\widehat{\mathbf{x}} \in \widehat{\mathbb{R}}^d$

$$(i = 0) \quad \widehat{\mathscr{L}_0}(\widehat{\mathbf{x}}) \overset{\text{def}}{=} 1 - \textstyle\sum_j \widehat{x}_j$$

$$(i \in [1..d]) \quad \widehat{\mathscr{L}_i}(\widehat{\mathbf{x}}) \overset{\text{def}}{=} \widehat{x}_{i-1}$$

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$$\forall i, j \in [0..d], \quad \widehat{\mathscr{L}_i}(\widehat{\mathbf{v}}_j) = \delta_{ij}$$

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$$\widehat{\boldsymbol{\mathscr{L}}} = \boldsymbol{\ell}^{\widehat{\mathbf{v}}} \overset{\text{def}}{=} \text{barycentric coordinates in } \widehat{\mathbf{v}}$$

$$\left(\text{i.e. } \sum_i \widehat{\mathscr{L}_i}(\widehat{\mathbf{x}})\widehat{\mathbf{v}}_i = \widehat{\mathbf{x}} \text{ with } \sum_i \widehat{\mathscr{L}_i} = 1\right)$$

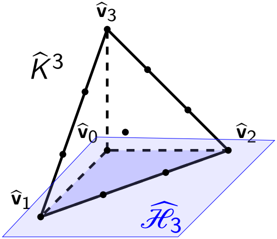
$$\forall i \in [0..d], \quad \widehat{\mathscr{L}_i} \text{ affine mapping}$$

$$\widehat{\boldsymbol{\mathscr{L}}} \text{ basis of } \mathbb{P}_1^d$$

Reference Simplex (3/3) — Face Hyperplanes

Reference face hyperplane

$$(i \in [0..d]) \quad \widehat{\mathscr{H}_i} \overset{\text{def}}{=} \text{span}(\widehat{\mathbf{v}}_j)_{j \neq i}$$



Reference Simplex (3/3) — Face Hyperplanes

Reference face hyperplane $(\rightsquigarrow \widehat{\mathscr{H}_i} = \text{ker}\, \widehat{\mathscr{L}_i})$

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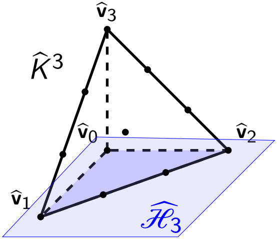

Reference Simplex (3/3) — Face Hyperplanes

Reference face hyperplane $(\rightsquigarrow \widehat{\mathscr{H}_i} = \ker \widehat{\mathscr{L}_i})$

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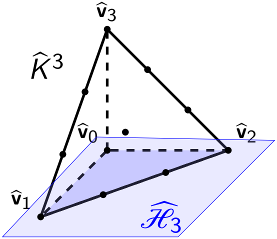

Ref injections: $\widehat{\mathbb{R}}^{d-1} \rightarrow \qquad \widehat{\mathbb{R}}^d$

$$(i = 0) \quad \widehat{I}_0(\widehat{\mathbf{x}}') \overset{\text{def}}{=} \left(1 - \textstyle\sum_j \widehat{x}'_j, \widehat{\mathbf{x}}'\right)$$

$$(i \in [1..d]) \quad \widehat{I}_i(\widehat{\mathbf{x}}') \overset{\text{def}}{=} (\widehat{x}'_0, \dots, \widehat{x}'_{i-2}, 0, \widehat{x}'_{i-1}, \dots, \widehat{x}'_{d-2})$$

Reference Simplex (3/3) — Face Hyperplanes

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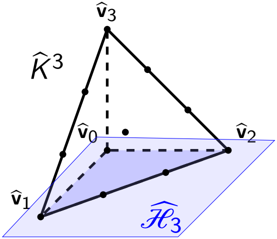

Reference Simplex (3/3) — Face Hyperplanes

Reference face hyperplane (⇝ $\widehat{\mathscr{H}}_i = \ker \widehat{\mathscr{L}}_i$)

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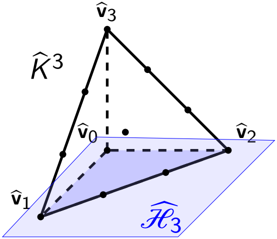
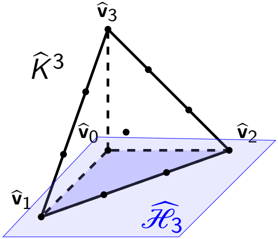


Hyperface multi-indices (⇝ $\boldsymbol{\alpha} \in \mathscr{A}^d_{k,i} \Leftrightarrow \widehat{\mathbf{a}}_{\boldsymbol{\alpha}} \in \widehat{\mathscr{H}}_i$)

$$\mathscr{A}^d_{k,0} \overset{\text{def}}{=} \{\boldsymbol{\alpha} \in \mathscr{A}^d_k \mid \sum_i \alpha_i = k\}, \qquad (i \in [1..d]) \quad \mathscr{A}^d_{k,i} \overset{\text{def}}{=} \{\boldsymbol{\alpha} \in \mathscr{A}^d_k \mid \alpha_{i-1} = 0\}$$

Reference Simplex (3/3) — Face Hyperplanes



Reference face hyperplane $\quad (\rightsquigarrow \widehat{\mathscr{H}}_i = \ker \widehat{\mathscr{L}}_i)$

$$(i \in [0..d]) \quad \widehat{\mathscr{H}}_i \overset{\text{def}}{=} \operatorname{span}(\widehat{\mathbf{v}}_j)_{j \neq i}$$

Ref injections: $\widehat{\mathbb{R}}^{d-1} \to \widehat{\mathscr{H}}_i \subset \widehat{\mathbb{R}}^d \quad (\rightsquigarrow \text{aff map})$

$$(i = 0) \quad \widehat{I}_0(\widehat{\mathbf{x}}') \overset{\text{def}}{=} \left(1 - \textstyle\sum_j \widehat{x}'_j, \widehat{\mathbf{x}}'\right)$$

$$(i \in [1..d]) \quad \widehat{I}_i(\widehat{\mathbf{x}}') \overset{\text{def}}{=} (\widehat{x}'_0, \dots, \widehat{x}'_{i-2}, 0, \widehat{x}'_{i-1}, \dots, \widehat{x}'_{d-2})$$

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Multi-index injections: $\mathscr{A}^{d-1}_k \to \mathscr{A}^d_{k,i} \subset \mathscr{A}^d_k$

$$(i = 0) \qquad \mathbb{I}^d_{k,0}(\boldsymbol{\alpha}') \overset{\text{def}}{=} (k - \textstyle\sum_j \alpha'_j, \boldsymbol{\alpha}')$$

$$(i \in [1..d]) \qquad \mathbb{I}^d_{k,i}(\boldsymbol{\alpha}') \overset{\text{def}}{=} (\alpha'_0, \dots, \alpha'_{i-2}, 0, \alpha'_{i-1}, \dots, \alpha'_{d-2})$$

Reference Simplex (3/3) — Face Hyperplanes

Reference face hyperplane $(\rightsquigarrow \widehat{\mathscr{H}}_i = \ker \widehat{\mathscr{L}}_i)$

$$(i \in [0..d]) \quad \widehat{\mathscr{H}}_i \overset{\text{def}}{=} \operatorname{span}(\widehat{\mathbf{v}}_j)_{j \neq i}$$

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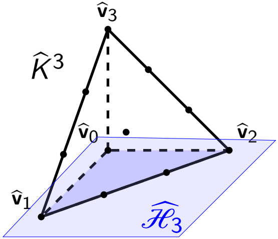

Hyperface multi-indices $(\rightsquigarrow \boldsymbol{\alpha} \in \mathscr{A}^d_{k,i} \Leftrightarrow \widehat{\mathbf{a}}_{\boldsymbol{\alpha}} \in \widehat{\mathscr{H}}_i)$

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Multi-index injections: $\mathscr{A}^{d-1}_k \to \mathscr{A}^d_{k,i} \subset \mathscr{A}^d_k$ $(\rightsquigarrow \widehat{I}_i(\widehat{\mathbf{a}}^{d-1}_{\boldsymbol{\alpha}'}) = \widehat{\mathbf{a}}^d_{\mathbb{I}^d_{k,i}(\boldsymbol{\alpha}')})$

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Outline

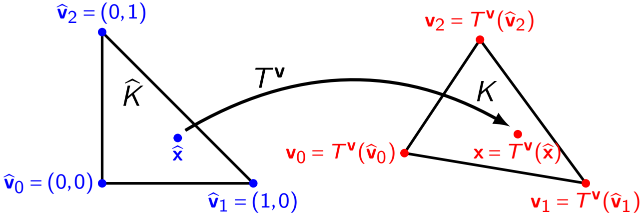
1. Introduction

2. About Measure Theory

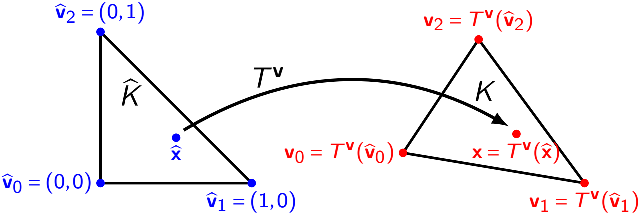
3. Simplicial Lagrange Finite Elements
    - Reference Simplex
    - Generic Simplex
    - Lagrange Finite Element

4. Difficulties, Conclusion and Perspectives

Generic Simplex (1/3) — Geometric Transformation



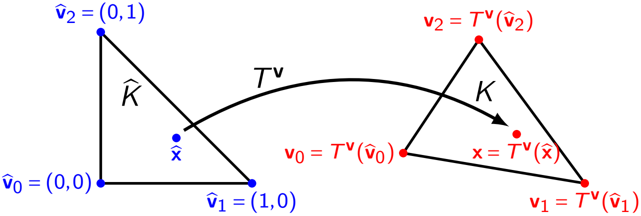
Generic Simplex (1/3) — Geometric Transformation



Geometric transformation given $d+1$ points $\mathbf{v}$ in $\mathbb{R}^d$

$$T^{\mathbf{v}}(\widehat{\mathbf{x}}) \overset{\text{def}}{=} \sum_{i\in[0..d]} \widehat{\mathscr{L}}_i(\widehat{\mathbf{x}})\mathbf{v}_i \qquad (T^{\mathbf{v}} : \widehat{\mathbb{R}}^d \to \mathbb{R}^d)$$

Generic Simplex (1/3) — Geometric Transformation



Geometric transformation given $d+1$ points $\mathbf{v}$ in $\mathbb{R}^d$

$$T^{\mathbf{v}}(\widehat{\mathbf{x}}) \overset{\text{def}}{=} \text{barycenter } (\widehat{\mathscr{L}}(\widehat{\mathbf{x}}), \mathbf{v}) \left(= \sum_{i \in [0..d]} \widehat{\mathscr{L}_i}(\widehat{\mathbf{x}}) \mathbf{v}_i \right) \qquad (T^{\mathbf{v}} : \widehat{\mathbb{R}}^d \to \mathbb{R}^d)$$

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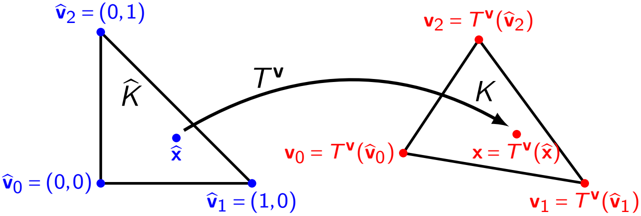

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Some properties

$T^{\mathbf{v}}$ affine mapping, $T^{\mathbf{v}}(\widehat{\mathbf{v}}) = \mathbf{v}$, $T^{\mathbf{v}}(\widehat{K}) = K$

v aff indep $\Longrightarrow$ $T^{\mathbf{v}}$ bij from $\widehat{\mathbb{R}}^d$ $(\widehat{K})$ onto $\mathbb{R}^d$ (K), $(T^{\mathbf{v}})^{-1}$ aff map

Generic Simplex (2/3) — Nodes and Face Hyperplanes

Nodes $\mathbf{a}^{\mathbf{v}}_{\boldsymbol{\alpha}} \overset{\text{def}}{=} T^{\mathbf{v}}(\widehat{\mathbf{a}}_{\boldsymbol{\alpha}})$ ($\leadsto$ $\mathbf{v}$ are nodes, $k = 1 \Rightarrow \mathbf{a}^{\mathbf{v}} =$ vertices)

Generic Simplex (2/3) — Nodes and Face Hyperplanes

Nodes $\mathbf{a}^{\mathbf{v}}_{\boldsymbol{\alpha}} \overset{\text{def}}{=} T^{\mathbf{v}}(\widehat{\mathbf{a}}_{\boldsymbol{\alpha}})$ ($\rightsquigarrow$ $\mathbf{v}$ are nodes, $k = 1 \Rightarrow \mathbf{a}^{\mathbf{v}} =$ vertices)

Lagrange polynomials $\mathbf{v} \in (\mathbb{R}^d)^{d+1}$ affinely independent

$$(i \in [0..d]) \quad \forall \mathbf{x} \in \mathbb{R}^d, \quad \mathcal{L}^{\mathbf{v}}_i(\mathbf{x}) \overset{\text{def}}{=} \widehat{\mathcal{L}_i} \circ (T^{\mathbf{v}})^{-1}(\mathbf{x})$$

$(T^{\mathbf{v}})^{-1}$ aff map $\rightsquigarrow$ $\mathcal{L}^{\mathbf{v}}_i(\mathbf{v}_j) = \delta_{ij}$, $\sum_i \mathcal{L}^{\mathbf{v}}_i = 1$, $\boldsymbol{\mathcal{L}}^{\mathbf{v}} = \boldsymbol{\ell}^{\mathbf{v}}$, $\mathcal{L}^{\mathbf{v}}_i$ aff map
$\boldsymbol{\mathcal{L}}^{\mathbf{v}}$ basis of $\mathbb{P}^d_1$

Generic Simplex (2/3) — Nodes and Face Hyperplanes

Nodes $\mathbf{a}^{\mathbf{v}}_{\boldsymbol{\alpha}} \overset{\text{def}}{=} T^{\mathbf{v}}(\widehat{\mathbf{a}}_{\boldsymbol{\alpha}})$ ($\rightsquigarrow$ $\mathbf{v}$ are nodes, $k = 1 \Rightarrow \mathbf{a}^{\mathbf{v}} =$ vertices)

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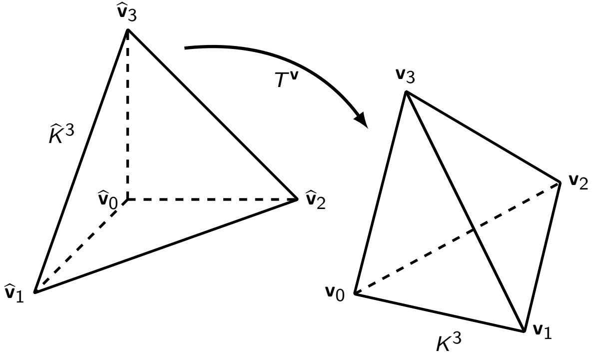
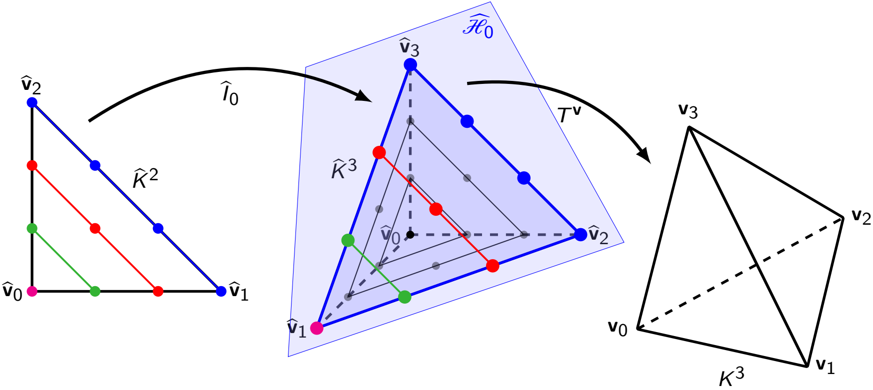
Face hyperplane $\mathscr{H}^{\mathbf{v}}_i \overset{\text{def}}{=} T^{\mathbf{v}}(\widehat{\mathscr{H}_i})$ ($\rightsquigarrow$ $\mathscr{H}^{\mathbf{v}}_i = \operatorname{span}(\mathbf{v}_j)_{j \neq i}$)

$\mathbf{v}$ aff indep $\Longrightarrow$ $\mathscr{H}^{\mathbf{v}}_i = \ker \mathscr{L}^{\mathbf{v}}_i$
$T^{\mathbf{v}}$ bij from $\widehat{\mathscr{H}_i}$ onto $\mathscr{H}^{\mathbf{v}}_i$
$\boldsymbol{\alpha} \in \mathscr{A}^d_{k,i} \Leftrightarrow \mathbf{a}^{\mathbf{v}}_{\boldsymbol{\alpha}} \in \mathscr{H}^{\mathbf{v}}_i$

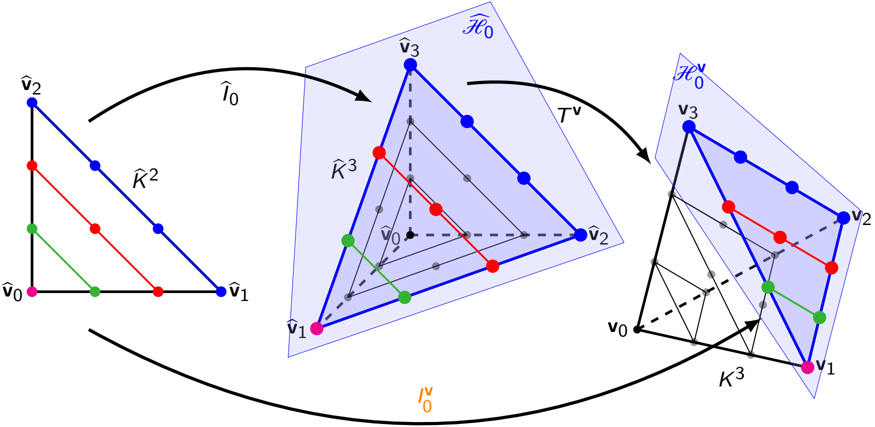
Generic Simplex (3/3) — Injections



Generic Simplex (3/3) — Injections



Generic Simplex (3/3) — Injections



Injections $\quad I_i^{\mathbf{v}} \stackrel{\text{def}}{=} T^{\mathbf{v}} \circ \widehat{I}_i \quad$ (aff map from $\widehat{\mathbb{R}}^{d-1}$ to $\mathbb{R}^d$)

$$\forall \boldsymbol{\alpha}' \in \mathscr{A}_k^{d-1}, \quad I_i^{\mathbf{v}}(\widehat{\mathbf{a}}^{d-1}_{\boldsymbol{\alpha}'}) = \mathbf{a}^{\mathbf{v}}_{\mathbb{I}^d_{k,i}(\boldsymbol{\alpha}')}$$

$\mathbf{v}$ aff indep $\implies$ $I_i^{\mathbf{v}}$ bij from $\widehat{\mathbb{R}}^{d-1}$ onto $\mathscr{H}_i^{\mathbf{v}}$

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Lagrange FE (1/5) — (K, P, Σ) Triple

Triple given $d, k \in \mathbb{N}$, $\mathbf{v} \in \left(\mathbb{R}^d\right)^{d+1}$

$$K \overset{\text{def}}{=} \operatorname{conv}(\mathbf{v})$$
$$P \overset{\text{def}}{=} \mathbb{P}_k^d \overset{\text{def}}{=} \operatorname{span}(\mathbf{X}^{\boldsymbol{\alpha}})_{\boldsymbol{\alpha} \in \mathscr{A}_k^d}$$
$$\boldsymbol{\Sigma} \overset{\text{def}}{=} \left(p \mapsto p(\mathbf{a}_{\boldsymbol{\alpha}}^{\mathbf{v}})\right)_{\boldsymbol{\alpha} \in \mathscr{A}_k^d}$$

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$$\boldsymbol{\Sigma} \overset{\text{def}}{=} (p \mapsto p(\mathbf{a}_{\boldsymbol{\alpha}}^{\mathbf{v}}))_{\boldsymbol{\alpha} \in \mathscr{A}_k^d} \qquad \rightsquigarrow \ \text{lin map}$$

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Unisolvence property $\left(\Phi_{\boldsymbol{\Sigma}}(p) \overset{\text{def}}{=} (p(\mathbf{a}_{\boldsymbol{\alpha}}^{\mathbf{v}}))_{\boldsymbol{\alpha} \in \mathscr{A}_k^d}\right)$

$$(d, k > 0) \quad \mathscr{U}(d,k) \overset{\text{def}}{=} \left[\forall \mathbf{v} \in (\mathbb{R}^d)^{d+1} \text{ aff indep, } \Phi_{\boldsymbol{\Sigma}} \text{ inj on } \mathbb{P}_k^d\right]$$

Lagrange FE (1/5) — (K, P, Σ) Triple

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Unisolvence property $(\Phi_{\boldsymbol{\Sigma}}(p) \overset{\text{def}}{=} (p(\mathbf{a}_{\boldsymbol{\alpha}}^{\mathbf{v}}))_{\boldsymbol{\alpha} \in \mathscr{A}_k^d})$

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Proof: double induction for $d, k \geq 1$

$$(k = 1, \mathbf{a}^{\mathbf{v}} = \mathbf{v}) \qquad p(\mathbf{x}) = \sum_{i \in [0..d]} p(v_i) \mathscr{L}_i^{\mathbf{v}}(\mathbf{x}) \qquad \rightsquigarrow \mathscr{U}(d, 1)$$

Lagrange FE (1/5) — (K, P, Σ) Triple

Triple given $d, k \in \mathbb{N}$, $\mathbf{v} \in (\mathbb{R}^d)^{d+1}$

$$\begin{aligned}
&K \overset{\text{def}}{=} \operatorname{conv}(\mathbf{v}) \\
&P \overset{\text{def}}{=} \mathbb{P}_k^d \overset{\text{def}}{=} \operatorname{span}(\mathbf{X}^{\boldsymbol{\alpha}})_{\boldsymbol{\alpha} \in \mathscr{A}_k^d} && \rightsquigarrow \dim P = \operatorname{card}(\mathscr{A}_k^d) = n_{\text{dof}} \\
&\mathbf{\Sigma} \overset{\text{def}}{=} (p \mapsto p(\mathbf{a}_{\boldsymbol{\alpha}}^{\mathbf{v}}))_{\boldsymbol{\alpha} \in \mathscr{A}_k^d} && \rightsquigarrow \text{lin map}
\end{aligned}$$

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$$(d = 1,\ \mathbf{a} \text{ pwd}) \qquad p(x) = \sum_{i \in [0..k]} p(a_i) \prod_{j \neq i} \frac{x - a_j}{a_i - a_j} \qquad \rightsquigarrow \mathscr{U}(1,k)$$

Lagrange FE (1/5) — (K, P, Σ) Triple

Triple given $d, k \in \mathbb{N}$, $\mathbf{v} \in (\mathbb{R}^d)^{d+1}$

$$\begin{array}{ll} K \overset{\text{def}}{=} \operatorname{conv}(\mathbf{v}) & \\ P \overset{\text{def}}{=} \mathbb{P}_k^d \overset{\text{def}}{=} \operatorname{span}(\mathbf{X}^{\boldsymbol{\alpha}})_{\boldsymbol{\alpha} \in \mathscr{A}_k^d} & \rightsquigarrow \dim P = \operatorname{card}(\mathscr{A}_k^d) = n_{\text{dof}} \\ \boldsymbol{\Sigma} \overset{\text{def}}{=} (p \mapsto p(\mathbf{a}_{\boldsymbol{\alpha}}^{\mathbf{v}}))_{\boldsymbol{\alpha} \in \mathscr{A}_k^d} & \rightsquigarrow \text{lin map} \end{array}$$

Unisolvence property $\left(\Phi_{\boldsymbol{\Sigma}}(p) \overset{\text{def}}{=} (p(\mathbf{a}_{\boldsymbol{\alpha}}^{\mathbf{v}}))_{\boldsymbol{\alpha} \in \mathscr{A}_k^d}\right)$

$$(d, k > 0) \quad \mathscr{U}(d,k) \overset{\text{def}}{=} \left[\forall \mathbf{v} \in (\mathbb{R}^d)^{d+1} \text{ aff indep, } \Phi_{\boldsymbol{\Sigma}} \text{ inj on } \mathbb{P}_k^d\right]$$

Proof: double induction for $d, k \geq 1$

$$(k = 1,\ \mathbf{a}^{\mathbf{v}} = \mathbf{v}) \qquad p(\mathbf{x}) = \sum_{i \in [0..d]} p(v_i) \mathscr{L}_i^{\mathbf{v}}(\mathbf{x}) \qquad \rightsquigarrow \mathscr{U}(d, 1)$$

$$(d = 1,\ \mathbf{a} \text{ pwd}) \qquad p(x) = \sum_{i \in [0..k]} p(a_i) \prod_{j \neq i} \frac{x - a_j}{a_i - a_j} \qquad \rightsquigarrow \mathscr{U}(1, k)$$

$\rightsquigarrow$ Show $\mathscr{U}(d+1, k) \quad \wedge \quad \mathscr{U}(d, k+1) \quad \Longrightarrow \quad \mathscr{U}(d+1, k+1)$

Lagrange FE (1/5) — (K, P, Σ) Triple

Triple given $d, k \in \mathbb{N}$, $\mathbf{v} \in (\mathbb{R}^d)^{d+1}$

$$K \overset{\text{def}}{=} \operatorname{conv}(\mathbf{v})$$
$$P \overset{\text{def}}{=} \mathbb{P}_k^d \overset{\text{def}}{=} \operatorname{span}(\mathbf{X}^{\boldsymbol{\alpha}})_{\boldsymbol{\alpha} \in \mathscr{A}_k^d} \qquad \rightsquigarrow \ \dim P = \operatorname{card}(\mathscr{A}_k^d) = n_{\text{dof}}$$
$$\boldsymbol{\Sigma} \overset{\text{def}}{=} (p \mapsto p(\mathbf{a}_{\boldsymbol{\alpha}}^{\mathbf{v}}))_{\boldsymbol{\alpha} \in \mathscr{A}_k^d} \qquad \rightsquigarrow \ \text{lin map}$$

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$\rightsquigarrow$ Show $\mathscr{U}(d+1,k) \ \wedge \ \mathscr{U}(d,k+1) \implies \mathscr{U}(d+1,k+1)$

Degenerate cases $d = 0$ and $k = 0$ are treated separately

Lagrange FE (2/5) — Polynomial Factorization

Lemma (factor zero on $\widehat{\mathscr{H}_d}$)

$$\forall p \in \mathbb{P}^d_{k+1}, \quad p_{|\widehat{\mathscr{H}_d}} = 0 \quad \Longrightarrow \quad \exists q \in \mathbb{P}^d_k,\, p = \widehat{\mathscr{L}_d} * q$$

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Lemma (factor zero on $\widehat{\mathscr{H}}_d$)

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Proof uses Euclidean division by monomial $X_{d-1} (= \widehat{\mathscr{L}}_d)$

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Lemma (factor zero on $\mathscr{H}^{\mathbf{v}}_i$)

$$\mathbf{v} \textit{ aff indep} \quad \Longrightarrow \quad \forall p \in \mathbb{P}^d_{k+1}, \quad p_{|\mathscr{H}^{\mathbf{v}}_i} = 0 \quad \Longrightarrow \quad \exists q \in \mathbb{P}^d_k,\ p = \mathscr{L}^{\mathbf{v}}_i * q$$

Lagrange FE (2/5) — Polynomial Factorization

Lemma (factor zero on $\widehat{\mathscr{H}}_d$)

$$\forall p \in \mathbb{P}^d_{k+1}, \quad p_{|\widehat{\mathscr{H}}_d} = 0 \quad \Longrightarrow \quad \exists q \in \mathbb{P}^d_k,\ p = \widehat{\mathscr{L}}_d * q$$

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Proof.

$\widehat{p} \overset{\text{def}}{=} p \circ T^{\tau^i_d(\mathbf{v})}$

$\square$

Lagrange FE (2/5) — Polynomial Factorization

Lemma (factor zero on $\widehat{\mathcal{H}}_d$)

$$\forall p \in \mathbb{P}^d_{k+1}, \quad p_{|\widehat{\mathcal{H}}_d} = 0 \quad \Longrightarrow \quad \exists q \in \mathbb{P}^d_k, \, p = \widehat{\mathcal{L}}_d * q$$

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Lemma (factor zero on $\mathcal{H}^{\mathbf{v}}_i$)

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Proof.

$\widehat{p} \stackrel{\text{def}}{=} p \circ T^{\tau^i_d(\mathbf{v})} \in \mathbb{P}^d_{k+1}$ (since $T^{\tau^i_d(\mathbf{v})}$ aff map)

□

Lagrange FE (2/5) — Polynomial Factorization

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$\rightsquigarrow \widehat{p}_{|\widehat{\mathcal{H}}_d} = 0$ (since $T^{\tau^i_d(\mathbf{v})}(\widehat{\mathcal{H}}_d) = \mathcal{H}^{\mathbf{v}}_{\tau^i_d(d)} = \mathcal{H}^{\mathbf{v}}_i$)

☐

Lagrange FE (2/5) — Polynomial Factorization

Lemma (factor zero on $\widehat{\mathcal{H}}_d$)

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Proof.

$\widehat{p} \overset{\text{def}}{=} p \circ T^{\tau'^{i}_{d}(\mathbf{v})} \in \mathbb{P}^d_{k+1}$ (since $T^{\tau'^{i}_{d}(\mathbf{v})}$ aff map)

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□

Lagrange FE (2/5) — Polynomial Factorization

Lemma (factor zero on $\widehat{\mathscr{H}}_d$)

$$\forall p \in \mathbb{P}^d_{k+1}, \quad p_{|\widehat{\mathscr{H}}_d} = 0 \quad \Longrightarrow \quad \exists q \in \mathbb{P}^d_k,\ p = \widehat{\mathscr{L}}_d * q$$

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$\rightsquigarrow p = \widehat{p} \circ \left(T^{\tau^i_d(\mathbf{v})}\right)^{-1}$

☐

Lagrange FE (2/5) — Polynomial Factorization

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Proof.

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$\rightsquigarrow p = \widehat{p} \circ (T^{\tau'^{i}_{d}(\mathbf{v})})^{-1} = \widehat{\mathscr{L}}_d \circ (T^{\tau'^{i}_{d}(\mathbf{v})})^{-1} * \widehat{q} \circ (T^{\tau'^{i}_{d}(\mathbf{v})})^{-1}$

□

Lagrange FE (2/5) — Polynomial Factorization

Lemma (factor zero on $\widehat{\mathscr{H}}_d$)

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Proof.

$\widehat{p} \overset{\text{def}}{=} p \circ T^{\tau'^{i}_{d}(\mathbf{v})} \in \mathbb{P}^d_{k+1}$ (since $T^{\tau'^{i}_{d}(\mathbf{v})}$ aff map)

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□

Lagrange FE (2/5) — Polynomial Factorization

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$q \overset{\text{def}}{=} \widehat{q} \circ (T^{\tau'^i_d(\mathbf{v})})^{-1} \in \mathbb{P}^d_k$ (since $(T^{\tau'^i_d(\mathbf{v})})^{-1}$ aff map) $\rightsquigarrow p = \mathscr{L}^{\mathbf{v}}_i * q$ ☐

Lagrange FE (3/5) — Unisolvence Proof

$$(d,k>0) \qquad \mathscr{U}(d+1,k) \quad \wedge \quad \mathscr{U}(d,k+1) \quad \Longrightarrow \quad \mathscr{U}(d+1,k+1)$$

Hyp: $\mathbf{v} \in (\mathbb{R}^{d+1})^{d+2}$ aff indep

$p \in \mathbb{P}^{d+1}_{k+1}$ st $\forall \boldsymbol{\alpha} \in \mathscr{A}^{d+1}_{k+1},\ p(\mathbf{a}^{\mathbf{v},d+1}_{\boldsymbol{\alpha}}) = 0$

Show: $p = 0$

Lagrange FE (3/5) — Unisolvence Proof

$$(d,k>0) \qquad \mathscr{U}(d+1,k) \quad \wedge \quad \mathscr{U}(d,k+1) \quad \Longrightarrow \quad \mathscr{U}(d+1,k+1)$$

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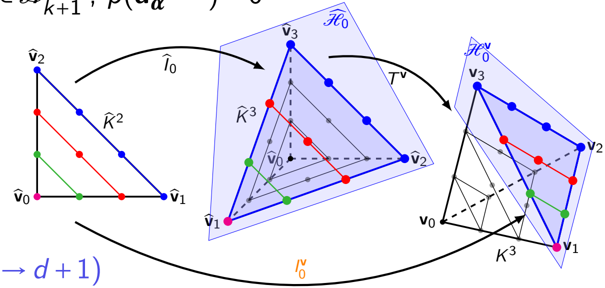

Step 1: factorization ($d \to d+1$)

$$\widehat{p}_0 \overset{\text{def}}{=} p \circ I^{\mathbf{v}}_0 : \widehat{\mathbb{R}}^d \to \mathscr{H}^{\mathbf{v}}_0 \subset \mathbb{R}^{d+1} \to \mathbb{R}$$

Lagrange FE (3/5) — Unisolvence Proof

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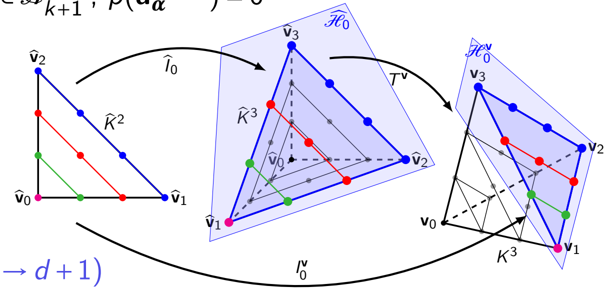

Step 1: factorization ($d \to d+1$)

$\widehat{p}_0 \overset{\text{def}}{=} p \circ I_0^{\mathbf{v}} : \widehat{\mathbb{R}}^d \to \mathscr{H}_0^{\mathbf{v}} \subset \mathbb{R}^{d+1} \to \mathbb{R}$

$\rightsquigarrow \widehat{p}_0 \in \mathbb{P}^d_{k+1}$ ($I_0^{\mathbf{v}}$ aff map)

Lagrange FE (3/5) — Unisolvence Proof

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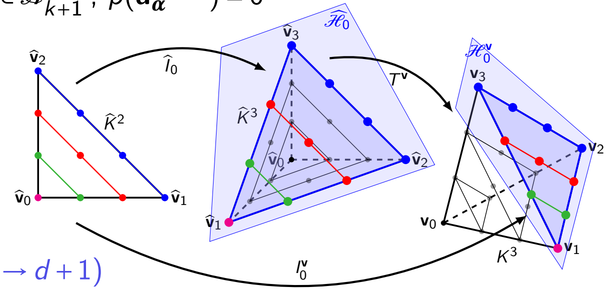

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$\rightsquigarrow\ \widehat{p}_0 \in \mathbb{P}^d_{k+1}$ ($I^{\mathbf{v}}_0$ aff map), and $I^{\mathbf{v}}_0(\widehat{\mathbf{a}}^d_{\boldsymbol{\alpha}'}) = \mathbf{a}^{\mathbf{v},d+1}_{\mathbb{I}^d_{k,i}(\boldsymbol{\alpha}')} \in \mathscr{H}^{\mathbf{v}}_0$ $(\boldsymbol{\alpha}' \in \mathscr{A}^d_{k+1})$

Lagrange FE (3/5) — Unisolvence Proof

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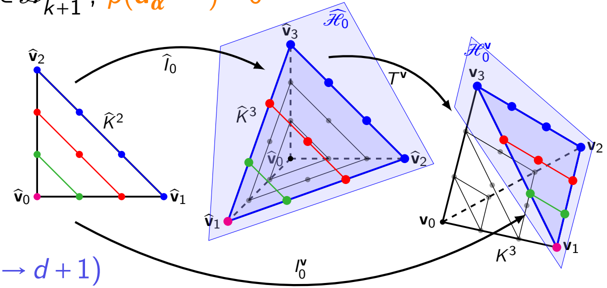

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Lagrange FE (3/5) — Unisolvence Proof

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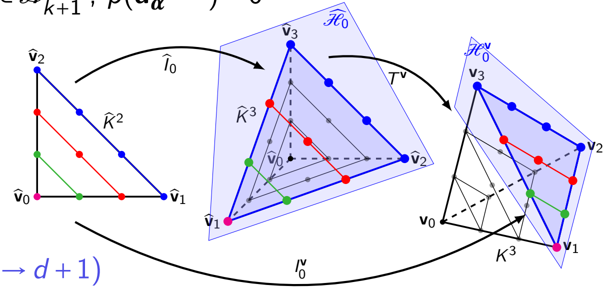

Step 1: factorization $(d \to d+1)$

$\widehat{p}_0 \overset{\text{def}}{=} p \circ I_0^{\mathbf{v}} : \widehat{\mathbb{R}}^d \to \mathscr{H}_0^{\mathbf{v}} \subset \mathbb{R}^{d+1} \to \mathbb{R}$

$\rightsquigarrow$ $\widehat{p}_0 \in \mathbb{P}^d_{k+1}$ ($I_0^{\mathbf{v}}$ aff map), and $I_0^{\mathbf{v}}(\widehat{\mathbf{a}}^d_{\boldsymbol{\alpha}'}) = \mathbf{a}^{\mathbf{v},d+1}_{\mathbb{I}^d_{k,i}(\boldsymbol{\alpha}')} \in \mathscr{H}_0^{\mathbf{v}}$ $(\boldsymbol{\alpha}' \in \mathscr{A}^d_{k+1})$

$\rightsquigarrow$ $\widehat{p}_0(\widehat{\mathbf{a}}^d_{\boldsymbol{\alpha}'}) = p(\mathbf{a}^{\mathbf{v},d+1}_{\mathbb{I}^d_{k,i}(\boldsymbol{\alpha}')}) = 0$ $\rightsquigarrow$ $\widehat{p}_0 = 0$ on $\widehat{\mathbb{R}}^d$

Lagrange FE (3/5) — Unisolvence Proof

$$(d, k > 0) \qquad \mathscr{U}(d+1, k) \quad \wedge \quad \mathscr{U}(d, k+1) \quad \Longrightarrow \quad \mathscr{U}(d+1, k+1)$$

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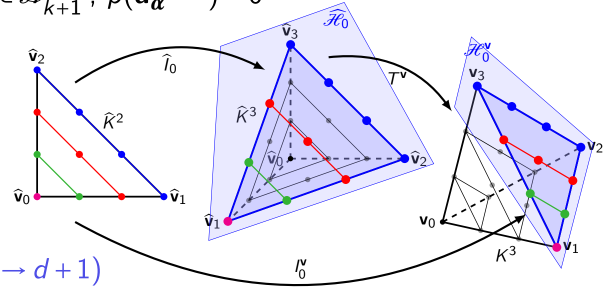


Step 1: factorization ($d \to d+1$)

$\widehat{p}_0 \overset{\text{def}}{=} p \circ I^{\mathbf{v}}_0 : \widehat{\mathbb{R}}^d \to \mathscr{H}^{\mathbf{v}}_0 \subset \mathbb{R}^{d+1} \to \mathbb{R}$

$\rightsquigarrow \widehat{p}_0 \in \mathbb{P}^d_{k+1}$ ($I^{\mathbf{v}}_0$ aff map), and $I^{\mathbf{v}}_0(\widehat{\mathbf{a}}^d_{\boldsymbol{\alpha}'}) = \mathbf{a}^{\mathbf{v},d+1}_{\mathbb{I}^d_{k,i}(\boldsymbol{\alpha}')} \in \mathscr{H}^{\mathbf{v}}_0$ ($\boldsymbol{\alpha}' \in \mathscr{A}^d_{k+1}$)

$\rightsquigarrow \widehat{p}_0(\widehat{\mathbf{a}}^d_{\boldsymbol{\alpha}'}) = p(\mathbf{a}^{\mathbf{v},d+1}_{\mathbb{I}^d_{k,i}(\boldsymbol{\alpha}')}) = 0 \rightsquigarrow \widehat{p}_0 = 0$ on $\widehat{\mathbb{R}}^d$

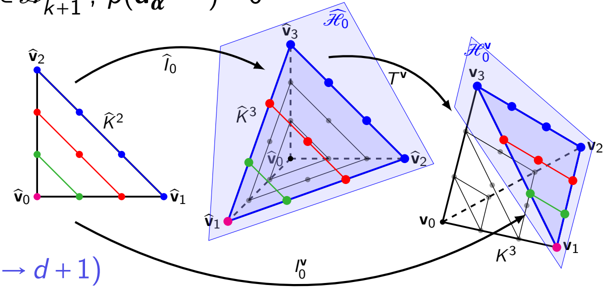
$p_{|\mathscr{H}^{\mathbf{v}}_0} = \widehat{p}_0 \circ (I^{\mathbf{v}}_0)^{-1} = 0$ ($I^{\mathbf{v}}_0$ bij)

Lagrange FE (3/5) — Unisolvence Proof

$$(d,k>0) \qquad \mathscr{U}(d+1,k) \quad \wedge \quad \mathscr{U}(d,k+1) \quad \Longrightarrow \quad \mathscr{U}(d+1,k+1)$$

Hyp: $\mathbf{v}\in(\mathbb{R}^{d+1})^{d+2}$ aff indep Show: $p=0$

$p\in\mathbb{P}^{d+1}_{k+1}$ st $\forall\boldsymbol{\alpha}\in\mathscr{A}^{d+1}_{k+1}$, $p(\mathbf{a}^{\mathbf{v},d+1}_{\boldsymbol{\alpha}})=0$



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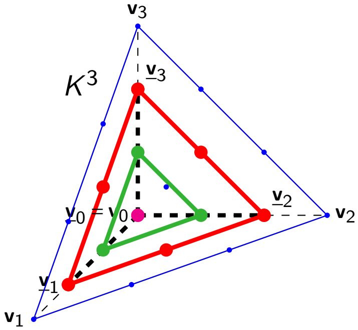

Step 2: cancellation ($k \to k+1$)

Subnodes: $\widetilde{\boldsymbol{\alpha}} \in \mathscr{A}^{d+1}_{k}$

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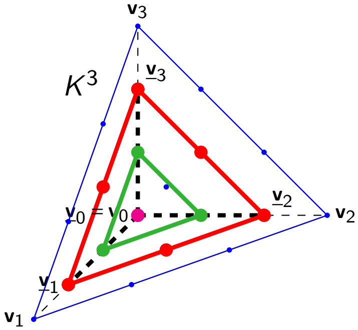

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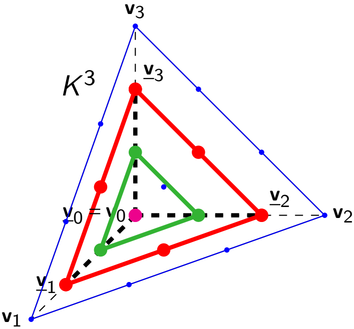

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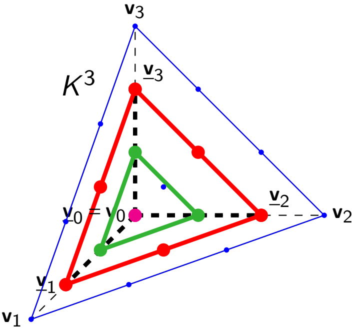

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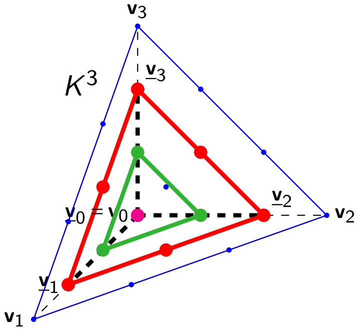

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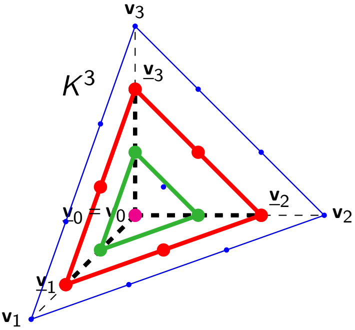

Lagrange FE (4/5) — Unisolvence Proof

$$(d, k > 0) \qquad \mathscr{U}(d+1, k) \quad \wedge \quad \mathscr{U}(d, k+1) \quad \implies \quad \mathscr{U}(d+1, k+1)$$

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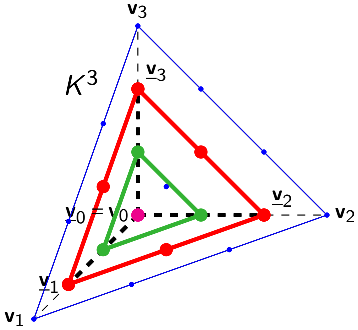

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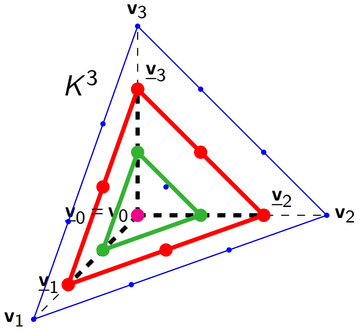

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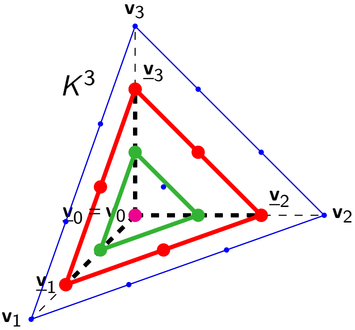

Note: reference subvertices $\widehat{\underline{\mathbf{v}}}$ are not reference vertices

$\rightsquigarrow$ proof does not work for the sole reference case

Lagrange FE (5/5) — Record Instantiation

```
Record FE (d : ℕ) : Type := mk_FE {
  nvtx : ℕ := d.+1;
  K_vertices : [0..nvtx) → 'ℝ^d;
  K_geom : subset 'ℝ^d := convex_hull K_vertices;
  ndof : ℕ;
  P_approx : subset ('ℝ^d → ℝ);
  P_approx_has_dim : has_dim P_approx ndof;
  S_dof : '(('ℝ^d → ℝ) → ℝ)^ndof;
  S_dof_lm : ∀ (i : [0..ndof)), lin_map (S_dof i);
  unisolvence_inj : KerS0 P_approx (gather S_dof);
}.

Context {d k : ℕ} (Hd : d ≠ 0) (Hk : k ≠ 0).
Context {vtx : [0..d] → 'ℝ^d} (Hvtx : aff_indep vtx).

Definition FE_LagPSdSk : FE d := mk_FE vtx (Pdk_has_dim d k)
  (FE_LagPSdSk_S_dof_lm k vtx) (FE_LagPSdSk_unisolvence_inj Hd Hk Hvtx).
```

Outline

Some Difficulties

Handling of vectors

even with (`H` : `n1` $=$ `n2`), `'T^n1` and `'T^n2` are not unifiable

$\rightsquigarrow$ explicit use of `castF H` : `'T^n1` $\rightarrow$ `'T^n2`

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Handling of algebraic substructures

even with `HPE` : `compatible_ms PE`, `PE` : `subset E` is not a `ModuleSpace`

⇝ explicitly build `PE_ms` := `sub_ModuleSpace HPE`

use `val` : `PE_ms` → `E` and `mk_sub` : (`Hx` : `PE x`) → `PE_ms`

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`T` → `E` inherits the algebraic structure of `E`

problem when several distinct algebraic structures match

e.g. `'R^d` : `AffineSpace R` masks `'R^d` : `ModuleSpace R`

⇝ some statement duplication to force `ModuleSpace` as `AffineSpace`

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Lack of analysis results

e.g. freedom of multivariate monomials $(\mathbf{X}^{\boldsymbol{\alpha}})_{\boldsymbol{\alpha}\in\mathscr{A}_k^d}$ without diff calculus

$\rightsquigarrow$ choose alternate proof paths using already formalized tools

Conclusion

- Formal tools provide high confidence in the proofs
  $\rightsquigarrow$ formalization of mathematics
  $\rightsquigarrow$ for properties of programs (including FP rounding error bounds)
- Formal definition of FE
- Formal build of simplicial Lagrange FE (+ subset/algebra: >45 kloc)
- Interdisciplinary work: mathematics / computer science
- Long-term work

More Details

Papers

Waves: https://hal.science/hal-00649240 (JAR, 2013)
https://hal.science/hal-00769201 (CAMWA, 2014)

Lax–Milgram: https://hal.science/hal-01344090 (Detailed proofs) (RR, 2016)
https://hal.science/hal-01391578 (CPP, 2017)

Lebesgue: https://hal.science/hal-03105815 (Detailed proofs) (RR, 2021)
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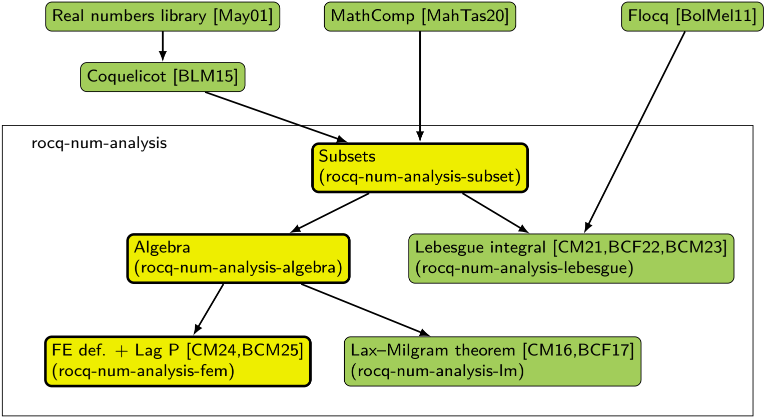
Rocq developments

Opam packages: https://rocq-prover.org/p/rocq-num-analysis/2.0.0
Git repo: https://lipn.univ-paris13.fr/rocq-num-analysis/

Perspectives



Real numbers library [May01]

MathComp [MahTas20]

Flocq [BolMel11]

Coquelicot [BLM15]

rocq-num-analysis

Subsets
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Algebra
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(rocq-num-analysis-lebesgue)

FE def. + Lag P [CM24,BCM25]
(rocq-num-analysis-fem)

Lax–Milgram theorem [CM16,BCF17]
(rocq-num-analysis-lm)

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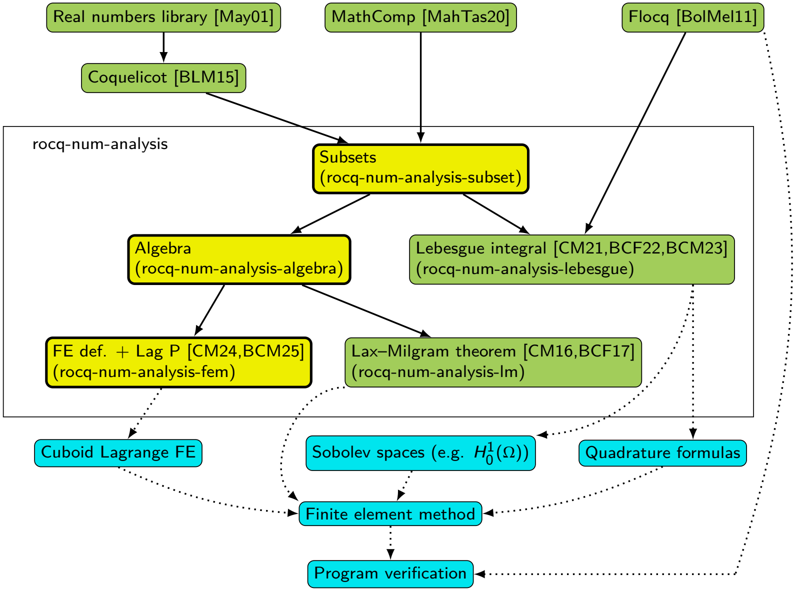

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Cuboid Lagrange FE

Sobolev spaces (e.g. $H_0^1(\Omega)$)

Quadrature formulas

Finite element method

Program verification

Thanks for your attention